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# The symmetry background underlying the ring structures of quantum dots and a classification scheme

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## Abstract

Due to the constraints arising from symmetry, each symmetric geometric configuration is accessible only to a specific group of states having specific quantum numbers. On the basis of the accessibility of the important symmetric configurations of a two-dimensional  $N$ -electron dot and/or of its subsystems, a scheme is proposed for classifying the states. In this scheme, each type of state has its own preference for ring structures. Numerical calculations for 9-electron and 19-electron dots have been performed to check the scheme. It was found that, on the basis of the scheme, the electronic structures of low-lying states can be predicted nearly perfectly. The symmetry background underlying the Laughlin wavefunction and the single-vortex states has also been discussed. Incidentally, an integer can be ultimately resolved as a product of prime numbers; this resolution has been used in the classification.

## 1. Introduction

The investigation of two-dimensional quantum dots with  $N$  electrons has been a hot topic in recent years [1–3]. When  $N$  is small, the effect of symmetry was found to be very important, e.g., the magic angular momenta of few-electron dots originate from the constraint of symmetry [4, 5]. When  $N$  is larger (say,  $N$  is close to or larger than 10), the effect of symmetry has scarcely been studied. The systems with larger  $N$  are themselves very attractive, because they might possess the features of few-body and of many-body systems. Thus the understanding of these systems might serve as a bridge to connect few-body and many-body physics. This paper is dedicated to the study of two-dimensional quantum dots with a larger  $N$ . From a preliminary study of these systems, shell structures have been found in some specific cases [6]. A more general picture of the dots would consist of a core surrounding by a ring [7–10]. At the present time, the constraints imposed by symmetry upon this ring model are not at all clear. It seems that an investigation of the underlying symmetry is necessary and timely.

The first aim of this paper is to study how symmetry would affect the electronic structures of the dots with a larger  $N$ . On the other hand, since many physical quantities in quantum

mechanics appear as integers, number theory might have a connection with the quantum mechanics. According to the ultimate resolution of integers, the orbital angular momentum  $L$  can be factorized as a product of prime numbers:  $L = 2^{n_2} 3^{n_3} 5^{n_5} 7^{n_7} \dots$ . Would this resolution lead to a classification of states? Our second aim is to look at this point. In what follows, an analysis based on symmetry will be made; then numerical results for nine- and nineteen-electron dots will be given.

## 2. Resolution of integers and a primary classification of states

Let us consider an eigenstate  $\Psi_{LS}$  of an  $N$ -electron dot, where  $L$  and the total spin  $S$  are good quantum numbers. Let  $\chi_i^{\tilde{\lambda}}$  be a spin state which is the  $i$ th component of the representation  $\tilde{\lambda}$  of the permutation group, where  $\tilde{\lambda}$  is a two-row representation with  $\frac{N}{2} + S$  blocks in the upper row and  $\frac{N}{2} - S$  blocks in the lower row.  $\Psi_{LS}$  can be expanded as

$$\Psi_{LS} = \sum_i F_i^\lambda \chi_i^{\tilde{\lambda}} \quad (1)$$

where  $F_i^\lambda$  is a function of spatial coordinates and is a basis state of the  $\lambda$ -representation, the representation conjugate to  $\tilde{\lambda}$ .

When  $N$  (or  $N - 1$ ) is a product of integers, i.e.,  $N = N_A N_B$  (or  $N - 1 = N_C N_D$ ), the electrons may form a symmetric geometric configuration with an  $m$ -fold axis,  $m = N_A, N_B, N_C$ , or  $N_D$ . In such a configuration there would be  $k$  homocentric circles each containing  $m$  electrons equidistantly distributed on the circle (some circles might have the same radius), where  $mk = N$  or  $(N - 1)$ . This configuration is simply denoted as  $m(k)$ . When  $mk = (N - 1)$ , there would be an electron at the centre. Some of the  $m(k)$  configurations are in the domain of low total potential energy; these  $m(k)$  are advantageous to binding and therefore important to low-lying states. Thus, if some or all of these favoured  $m(k)$  are prohibited by symmetry (as we shall see is the case), there would be serious consequences.

Since a rotation of an  $m(k)$  configuration about the centre by  $\frac{2\pi}{m}$  is equivalent to  $k$  cyclic permutations of particles, we have

$$e^{i2\pi L/m} F_i^\lambda(12\dots) = \sum_j G_{ji}^\lambda(p_c) F_j^\lambda(12\dots) \quad (2)$$

where  $p_c$  denotes the  $k$  cyclic permutations, and  $G_{ji}^\lambda(p_c)$  is the associated matrix element of the  $\lambda$ -representation. It is emphasized that this equation holds only if the particles form an  $m(k)$  configuration [5]. Such equations form a set of homogeneous linear equations with the determinant

$$D(L, \lambda, m, k) = |G_{ji}^\lambda(p_c) - \delta_{ij} e^{i2\pi L/m}|. \quad (3)$$

Evidently, if  $D$  is non-zero, the  $F_i^\lambda$  must all be zero at the  $m(k)$ . In this case, an inherent nodal surface is imposed by symmetry and the  $m(k)$  is therefore inaccessible [11]. Thus, whether  $D$  is non-zero or zero would affect strongly the electronic structure of the state.

When a magnetic field  $B$  vertical to the plane of the dot is applied, the total spin of the true ground state  $S_{gro}$  depends on  $B$ . There are separate regions of  $B$  having  $S_{gro}$  equal to  $N/2$  [12]. In particular, when  $B$  is large, we always have  $S_{gro} = N/2$ . In this paper we discuss only the case of  $S = N/2$ . In this case  $\lambda$  is totally antisymmetric and the determinant has a simpler form:

$$D(L, \lambda, m, k) = (-1)^{(m-1)k} - e^{i2\pi L/m} \quad (4)$$

It is assumed that  $L \geq 0$  (the case of  $L < 0$  is similar). Let a series of states having the same  $L$  be called an  $L$ -series; they are denoted as  $(L)_n$  ( $n = 1, 2, \dots$ ). For a given  $L$ , if a

couple of  $m$  and  $k$  lead to  $D(L, \lambda, m, k) = 0$ , then the corresponding  $m(k)$  configuration is accessible to the  $L$ -series of an  $N$ -electron system with  $N = mk$  or  $mk + 1$ . Evidently, it is clear from equation (4) that all the  $m(k)$  are accessible to the ( $L = 0$ )-series except that in the case of  $m$  even and  $k$  odd. When  $L \geq 1$ , from the resolution of integers we have

$$L = 2^{n_2} 3^{n_3} 5^{n_5} 7^{n_7} (11)^{n_{11}} \dots \tag{5a}$$

and

$$m = 2^{m_2} 3^{m_3} 5^{m_5} 7^{m_7} (11)^{m_{11}} \dots \tag{5b}$$

Then the discriminant  $D = 0$  reads

$$\exp[i\pi (2^{n_2+1-m_2} 3^{n_3-m_3} 5^{n_5-m_5} \dots)] = (-1)^{(m-1)k}. \tag{6}$$

For a given  $L$  and a given  $m(k)$  configuration, if equation (6) is fulfilled, then the  $m(k)$  is accessible to the  $L$ -series. From (6) we have the following rules:

- Rule 1. If  $m$  is odd, or if  $m$  and  $k$  are both even, then the  $m(k)$  is accessible to the  $L$ -series with  $n_i \geq m_i$  (here  $i = 2, 3, 5, \dots$ ).
- Rule 2. If  $m$  is even and  $k$  is odd, then the  $m(k)$  is accessible to the  $L$ -series with  $n_2 = m_2 - 1$  and  $n_i \geq m_i$  (here  $i = 3, 5, \dots$ ).
- Rule 3. If  $m = m_a m_b$  and the configuration  $m(k)$  is accessible to an  $L$ -series, then both the configurations  $m_a(km_b)$  and  $m_b(km_a)$  are accessible to the  $L$ -series; alternatively, if  $m_a(km_b)$  or  $m_b(km_a)$  is inaccessible to an  $L$ -series, the  $m(k)$  is also inaccessible to the  $L$ -series. When both  $m_a(km_b)$  and  $m_b(km_a)$  are accessible to an  $L$ -series, the  $m(k)$  would be definitely accessible to the  $L$ -series if  $m_a$  and  $m_b$  do not have a common factor.

In the case of  $k = 1$ , it is already known [4] that the  $m(1)$  configuration is accessible to the  $L = mj$  states ( $j$  denotes an integer throughout this paper) if  $m$  is odd, or to the  $L = \frac{m}{2} j_o$  states ( $j_o$  is an odd integer throughout this paper) if  $m$  is even. The above rules are a generalization of the  $k = 1$  case.

For an example of the application of the above rules, let us consider the 8(3) configuration. Since this configuration has  $m$  even and  $k$  odd, rule 2 is applicable. Since in this case  $m_2 = 3$  and  $m_i = 0$  ( $i \neq 2$ ), rule 2 indicates that the 8(3) is accessible to the  $L$ -series of an  $N = 24$  or 25 system with  $n_2 = 2$  and  $n_i \geq 0$  (i.e.,  $L = 4j_o$ ).

Let us consider whether the above rules are connected with some experimentally observed phenomena. From the above rules, it is straightforward to deduce the following lemma:

**Lemma.** *If  $I$  is an integer and  $L = I(I \pm j_o)/2 \geq 0$ , where  $j_o$  is an odd integer, then any  $m(k)$  configuration with  $mk = I$  is accessible to the  $L$ -series.*

From the lemma we know that all the  $m(k)$  with  $mk = N$  are accessible to the  $L = N(N - 1)/2 \pm jN$  states, and all the  $m(k)$  with  $mk = N - 1$  are accessible to the  $L = N(N - 1)/2 \pm j(N - 1)$  states. Therefore, all the  $m(k)$  disregarding  $mk = N$  or  $N - 1$  are accessible to the  $L = j_o N(N - 1)/2$  states. Since the famous Laughlin wavefunction [13, 14]

$$\psi_v = \left[ \prod_{i < j} (z_i - z_j)^{j_o} \right] \exp\left(-\frac{1}{4} \sum_j z_j^* z_j\right) \tag{7}$$

has  $L = j_o N(N - 1)/2$ , these special states can get access to any  $m(k)$  disregarding  $mk = N$  or  $N - 1$ . It is noted that the domain(s) in coordinate space with a lower total potential energy are preferred by low-lying states, and their wavefunctions are concentrated in these domain(s). If there are some inaccessible  $m(k)$  located in these domain(s), inherent nodal surfaces will appear and will cause an increase of energy. In contrast, since the Laughlin wavefunctions can

get access to any  $m(k)$ , no inherent nodal surfaces are contained in them (except when two electrons overlap). Thus their energies would be relatively low and therefore they are good approximations of ground states. It is well known that the Laughlin states have the filling factor  $\nu = \frac{1}{j_o} = 1, \frac{1}{3}, \frac{1}{5}, \dots$ . Thus these filling factors associated with the experimentally observed plateaus in the Hall resistance are related to the accessibility of symmetric configurations.

Furthermore, the so-called quasihole or single-vortex state [15, 16] has  $L = j_o N(N - 1)/2 + N$ . From the above rules it is easy to see that all the  $m(k)$  configuration with  $mk = N$  are accessible to this vortex state, while all the  $m(k)$  with  $mk = N - 1$  are inaccessible to it. This is an interesting point. It implies that, in order to avoid the appearance of inherent nodal surfaces, the vortex state would deny the  $m(k)$  configurations with  $mk = N - 1$ , i.e., no particle would stay at the centre. Accordingly, in an approximate wavefunction of this state it is appropriate to include the factor  $\prod_i (z_i - z_c)$ , where  $z_c$  is the coordinate of the centre of mass.

The  $m(k)$  accessibility is an important inherent feature of states and therefore can be used as an objective basis for classifying the states. For an example, let us investigate in detail a nine-electron dot [7, 8, 17] and see how the states can be classified according to their  $m(k)$  accessibility. The nine electrons can form the 9(1), 3(3), 8(1), 4(2), and 2(4) configurations. From the above rules we can understand which states can get access to which of these configurations. For example, the 3(3) has  $m$  odd,  $m_3 = 1$ , and other  $m_i = 0$ . Thus according to rule 1, the 3(3) is accessible to the  $L$ -series with  $n_3 \geq 1$  and other  $n_i \geq 0$ , i.e., accessible to the states with  $L = 3j$ . On the basis of the accessibility we can arrive at a very detailed classification, where the states able to get access to a specific group of configurations are grouped into the same type. However, a very detailed scheme may not be necessary. Since we are mostly interested in low-lying states, the configurations associated with a considerably higher potential energy are not important and therefore can be neglected. We shall consider a simpler classification scheme based on only important  $m(k)$  configurations as follows.

It has been suggested that some of the electrons of the dot would form a ring outside together with a core inside [7–10, 18, 19]. Let the structure having  $n$  electrons in the ring be called an  $n$ -ring structure. For a nine-electron dot, the optimal potential energies of the 7-ring and 8-ring structures are the lowest [20], that of the 6-ring is the second lowest, those for rings with  $n = 9$  or  $n \leq 5$  are considerably higher. Therefore, a classification scheme mainly for the low-lying states may take only the 8-, 7-, and 6-ring structures into account. In an 8-ring structure, the minimum of the potential energy is located at an optimal centred octagon (an 8(1) configuration), and the wavefunction is mainly distributed around this octagon. Evidently, for 8(1)-inaccessible states, the 8-ring structure is no longer a favourable choice, because the inherent node at the octagon would push the wavefunction away from the minimum of the potential energy, speed up the internal oscillation, and thus spoil the stability. In a 6-ring structure, there are three particles in the core; the minimum of the potential energy is located at a 3(3) configuration (with the two outer homocentric circles having the same radius). Therefore, for a 3(3)-inaccessible state, the 6-ring structure is not a favourable choice. It is undoubtedly the case that the choice of an 8-ring or a 6-ring structure depends strongly on the 8(1) and 3(3) accessibilities. Therefore, as a first step, we propose a classification scheme based on the 8(1) and 3(3) accessibilities as shown in table 1, where, by making use of rules 1 to 3, the  $L$ -series are classified into four types. In this step, the important 7-ring structure has not yet been taken into account. Since this structure is constrained by symmetry in a different way, we shall take it into account in the next step. Nonetheless, the classification in table 1 is model independent; it depends purely on symmetry. It is an objective basis for a more detailed classification.

**Table 1.** A primary classification of states of a polarized nine-electron dot according to the 8(1) and 3(3) accessibility. The  $m(k)$  configurations accessible to a given type are listed in the second column; the  $L$ -series belonging to the given type are listed in the third column (the  $\{n_i\}$  of the  $L$  are given, the  $n_i$  not listed in the table are arbitrary, or  $n_i \geq 0$ ).

Type	$m(k)$	$\{n_i\}$
1	8(1), 3(3)	$n_2 = 2, n_3 \geq 1$
2	8(1)	$n_2 = 2, n_3 = 0$
3	3(3)	$n_2 \neq 2, n_3 \geq 1$
4		$n_2 \neq 2, n_3 = 0$

### 3. A detailed classification scheme for a nine-electron dot

In the previous section the accessibility of the  $m(k)$  configurations has been studied. This accessibility is governed by the constraints originating from the equivalence of rotation of symmetric configurations and cyclic permutations (refer to equation (2)). This idea can be generalized to the configuration with  $N_{out}$  electrons located outside in a ring and with  $N_{in}$  electrons located inside in a core;  $N_{out} + N_{in} = N$ . Let the angular momenta of the ring and the core be denoted as  $L_{ring}$  and  $L_{core}$ , respectively. Of course, they are not good quantum numbers, and there are many choices for them. For the ring, when the electrons are equidistantly distributed on a circle (forming a regular polygon), the rotation by the angle  $2\pi/N_{out}$  is equivalent to a cyclic permutation. Thus, the regular polygon is accessible to the outward subsystem only if  $L_{ring} = jN_{out}$  (if  $N_{out}$  is odd) or  $L_{ring} = j_o N_{out}/2$  (if  $N_{out}$  is even), just like an  $m(k)$  configuration with  $m = N_{out}$  and  $k = 1$ . Similarly, we can consider the  $m'(k')$  accessibility of the subsystem in the core; here  $m'k' = N_{in}$  or  $N_{in} - 1$ . In this way, we can identify the favourable choices of  $L_{ring}$  and  $L_{core}$  (in these choices, symmetric configurations for the subsystems are accessible). With this in mind, let us consider the 7-ring structure of a nine-electron dot.

Since 7 and 2 do not have a common factor, the 7-ring structure is not subject to the overall constraints imposed by the  $m(k)$  accessibility, but is subject to partial constraints as discussed in the previous paragraph. For the ring,  $L_{ring} = 7j$  is a favourable choice. For the core, the configuration with the two electrons located equidistantly on either side of the centre can reduce the Coulomb repulsion and therefore is advantageous as regards binding. However, since this favourable configuration is in fact a 2(1) configuration, it would be prohibited if  $L_{core}$  is even. Consequently, the 7-ring structure would be a favourable choice only if  $L = 7j + j_o$ . Moreover, since the rotation energy of an electron is proportional to  $(l_i/r_i)^2$ , where  $l_i$  the angular momentum of the electron and  $r_i$  the radial distance, it is better for the electrons in the core to have a very small  $l_i$ ; otherwise the rotation energy will greatly increase. This implies that, for low-lying states, the total angular momentum would be mainly contributed by  $L_{ring}$  while  $L_{core}$  would be very small. From experience with our numerical calculation, as shown later, we can state that the 7-ring structure is no longer a favourable choice if  $j_o \geq 5$ . Thus we can make the statement (arising partially from symmetry considerations and partially from dynamics and experience) that the 7-ring structure is a favourable choice if

$$L = 7j + j'_o \quad (j'_o = 1 \text{ or } 3). \quad (8)$$

With the help of equation (8), each type in table 1 can be further classified into A and B; the type A has  $L = 7j + 1$  or  $L = 7j + 3$ , while  $L$  for type B takes other values. This more detailed classification is given in table 2, where the  $L$  belonging to a specific type are listed ( $36 \leq L \leq 73$ ). For example, type 3A contains the  $L = 45, 57, 66, \dots$  states, in which the 7- and 6-ring structures are not constrained by symmetry and therefore are both favourable

**Table 2.** A detailed classification of the  $L$ -series of a nine-electron dot ( $36 \leq L \leq 73$ ). If an  $n$ -ring structure is a favourable choice for a given type, the associated block is marked by the symbol  $\times$  (e.g., both the 7- and 6-ring structures are favourable choices for the type 3A).

Type	7-ring	8-ring	6-ring	$L$
1A	$\times$	$\times$	$\times$	36
2A	$\times$	$\times$		52
3A	$\times$		$\times$	45, 57, 66
4A	$\times$			38, 43, 50, 59, 64, 71, 73
1B		$\times$	$\times$	60
2B		$\times$		44, 68
3B			$\times$	39, 42, 48, 51, 54, 63, 69, 72
4B				37, 40, 41, 46, 47, 49, 53, 55, 56, 58, 61, 62, 65, 67, 70

choices, while the 8-ring is not because the 8(1) configuration is inaccessible to this type.

If we neglect the electronic structures and emphasize just the energy, then the above eight types can be grouped into three groups. The first group contains the first six types 1A to 2B, and the 7-ring and/or 8-ring structures are favourable choices for them; therefore this group is better for binding. The second group contains only the type 3B; only the 6-ring is a favourable choice. Since the 6-ring is higher in potential energy, the binding is relatively weak. The third group contains only the type 4B; all the 7-, 8-, and 6-ring structures are non-favourable choices. Therefore the binding of the third group is the weakest. It is expected that the levels of the ‘first-states’ (a ‘first-state’ is the lowest state of an  $L$ -series) of the first group should be relatively low, while those of the third group should be higher. In order to see how the energies and the electronic structures are related to the classification, we perform numerical calculations and analyse the results as follows.

#### 4. Numerical results

Let the Hamiltonian be

$$H = \sum_i \left[ \frac{p_i^2}{2m^*} + \frac{1}{2} m^* \omega_0^2 r_i^2 \right] + \frac{e^2}{4\pi\epsilon} \sum_{i>j} \frac{1}{r_{ij}} \quad (9)$$

where  $m^* = 0.067 m_e$  and  $\epsilon = 12.4$  are assumed (for GaAs dots).  $\omega_0$  arises from a parabolic confinement and/or from a magnetic field  $B$  (the well known linear term proportional to  $B$  is ignored because this term does not affect the electronic structures at all, and thus is not relevant to the following discussion).

The method used to solve the Schrödinger equation is basically the same as the one given by Manninen *et al* [16]; the Hamiltonian is diagonalized in a space spanned by antisymmetrized many-body basis functions  $\Phi_i$  for  $S = N/2$  states. The set  $\Phi_i$  are composed of the Darwin–Fock single-particle states  $\varphi_{uv}$  with adjustable frequency  $\omega_{eff}$ . The  $\varphi_{uv}$  has an eigenenergy  $(u + v + 1)\hbar\omega_{eff}$  and an angular momentum  $(u - v)\hbar$ ; the parameter  $\omega_{eff}$  will be optimized in the calculation. In particular, the states from several Landau levels are included. The two-dimensional Talmi–Moshinsky coefficients are used to facilitate the calculation of matrix elements [21]. In order to confine the number of basis functions, first of all the maximal values of  $u$  and  $v$  for the single-particle states have to be fixed. Then they can constitute a limited set of  $\Phi_i$  for a given  $L$ . It is recalled that the lowest Landau levels have  $K$ , the sum of the  $v$  for the  $N$  particles, equal to zero. In what follows, only the  $\Phi_i$  with  $K = 0, 1$ , and 2 are taken into account. In order to further confine the number of  $\Phi_i$ , they are arranged in



a sequence such that  $\langle \Phi_i | H | \Phi_i \rangle \leq \langle \Phi_{i+1} | H | \Phi_{i+1} \rangle$ . Evidently, in this sequence the  $\Phi_i$  with a very large index  $i$  are not important to low-lying states. We found that such an arrangement is very helpful for improving the results if only matrices of finite dimension can be dealt with. The diagonalization was first carried out in a given smaller space (with the index  $i$  smaller), then again in a larger space (i.e., the  $\Phi_i$  with larger  $i$  are added), and again repeatedly until a satisfactory convergence of the eigenenergies is achieved. We found that, for the low-lying states concerned in the calculation,  $i \leq 12\,000$  is sufficient for obtaining qualitatively accurate results.

To show the convergency, examples are given as follows.

- (a) For  $N = 9$ ,  $L = 51$ , and  $\hbar\omega_0 = 3$  meV, the maximal values of  $u$  and  $v$  are first given as 16 and 2, and  $K \leq 2$  is assumed. Accordingly, the total number of  $\Phi_i$  is 34 215. When the lowest 4000, 6000, and 8000 of them are adopted, the corresponding lowest eigenenergies are 250.75, 250.64, and 250.60 meV, respectively. Although the convergency of the energy is not very good, the correlated densities (as shown later) extracted from the corresponding eigenwavefunctions remain nearly unchanged. Since the emphasis of this paper is not on precise calculations but on the qualitative aspects, the results from the calculation with 8000 basis functions are sufficient for our purpose. Also, we found that, for the lowest eigenstate with the 8000 basis functions, the weights of the  $K = 0, 1$ , and 2 components are 0.879, 0.107, and 0.014. This implies that, even if the  $K \geq 3$  components are included, the effect on the wavefunctions would be very small. Therefore, it is not necessary to include them. Furthermore, let us define the weight of a  $u_0$ -component, which is the sum of the squares of the amplitudes of the  $\Phi_i$  having the maximal value  $u = u_0$ . Then, for this state, the weights of the  $u_0 = 14, 15$ , and 16 components are 0.0306, 0.0114, and 0.0003. This implies that the Darwin–Fock states with  $u \geq 17$  do not have to be taken into account.
- (b) For  $N = 19$ ,  $L = 183$ , and  $\hbar\omega_0 = 3$  meV, the maximal  $u$  and  $v$  are first given as 28 and 1, and  $K \leq 1$  is assumed. Accordingly, the total number of  $\Phi_i$  is 21 968. When the lowest 4000, 8000, and 12 000 of them are adopted, respectively, the corresponding lowest eigenenergies are 893.79, 893.68, and 893.64. Again, the convergency is not very good but is sufficient for our purpose.

In what follows, let us first give the calculated spectrum of a nine-electron dot. Let  $E_n(L)$  be the energy of the  $(L)_n$  state, the  $n$ th state of an  $L$ -series. If Coulomb repulsion is removed, the states with  $L \geq N(N-1)/2$  remain in the lowest Landau levels with the energy  $(L+N)\hbar\omega_0$ . Therefore we define

$$E_{coul}(L) = E_1(L) - (L+N)\hbar\omega_0 \quad (10)$$

which measures the effect of Coulomb repulsion on the first-states  $(L)_1$ . These quantities are given in figure 1. In general, the increase of  $L$  would lead to a larger size and therefore a weaker Coulomb repulsion, and accordingly a smaller  $E_{coul}$ . However, the curve of  $E_{coul}(L)$  is not smooth. When  $L$  takes some specific values  $L_{pl}$ , there are a number of small plateaus arising from the equality  $E_{coul}(L_{pl}) = E_{coul}(L_{pl} + 1)$ . For example, a plateau starts from  $L_{pl} = 45$  and ends at  $L = 46$ , and so on. In this case we have  $E_1(L_{pl} + 1) = E_1(L_{pl}) + \hbar\omega_0$ ; this implies that the  $(L_{pl} + 1)_1$  state is simply a c.m. excitation of the  $(L_{pl})_1$  state, and it also implies that the internal energy of the  $(L_{pl} + 1)$ -series is considerably higher. One might suggest that if an  $L_{pl}$  is in the first group of table 2 while  $L_{pl} + 1$  is in the third group, then a plateau might emerge because this case would lead to a great difference in internal energy. From table 2 we know that these  $L_{pl}$  are 45, 52, 57, 60, 64, and 66 (within the range  $44 \leq L \leq 72$ ). It turns out that each member of this series of  $L$ -values picked from table 2 is actually associated with



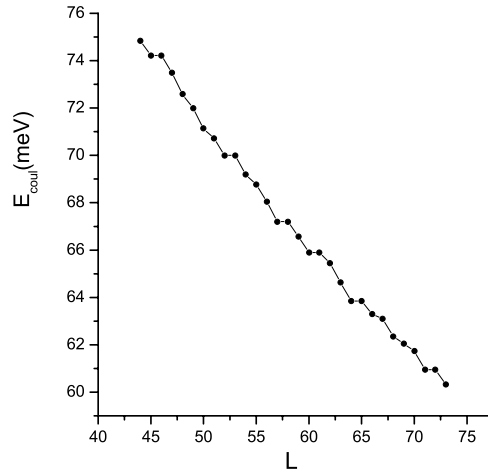


Figure 1.  $E_{coul}$  for a nine-electron dot as a function of  $L$ .  $\hbar\omega_0 = 3$  meV is assumed.

a plateau in figure 1 without exception (however, the plateau starting from  $L = 66$  is a little inclined). The inverse is also true, i.e., each plateau in figure 1 starts from an  $L$  belonging to the first group while  $L + 1$  belongs to the third group. The only exception to this inverse statement is the case of the plateau starting from  $L_{pl} = 71$  (the first group) and ending at 72 (the second group). The above facts demonstrate that the classification is closely related to the energies of states. In what follows, it will be shown that the classification is even more closely related to the electronic structures. Incidentally, the  $L_{pl}$ , namely  $L = 45, 52, 57 \dots$ , may be magic angular momenta associated with the true ground states [4]. Also, Ruan *et al* have used a different method based on hyperspherical harmonic coordinates [17] to solve the Schrödinger equation; the features of the spectra that they found were qualitatively the same as our findings (refer to figures 1(a) and 2(a) of the [17]).

The electronic structures of an eigenstate  $\Psi_{LS}$  can be demonstrated via the two-body and three-body density functions defined as

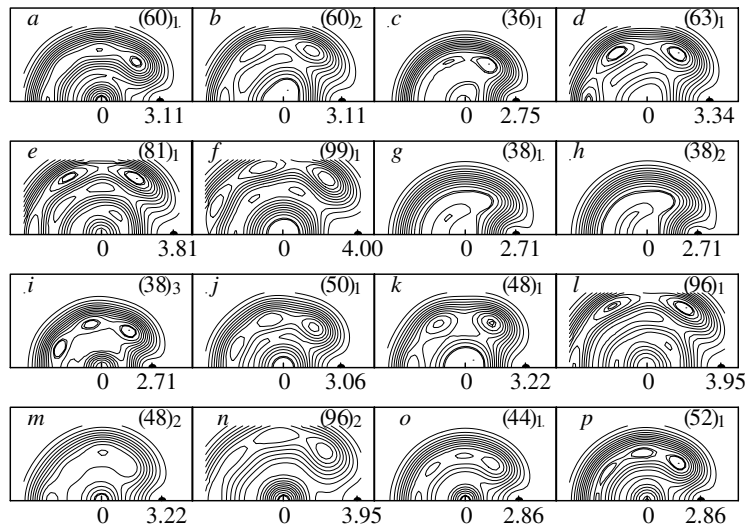
$$\rho_2(\vec{r}_1, \vec{r}_2) = \int d\vec{r}_3 \cdots d\vec{r}_9 |\Psi_{LS}|^2 \quad (11a)$$

and

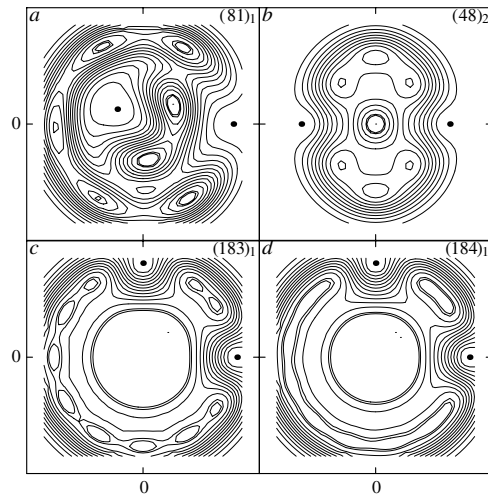
$$\rho_3(\vec{r}_1, \vec{r}_2, \vec{r}_3) = \int d\vec{r}_4 \cdots d\vec{r}_9 |\Psi_{LS}|^2. \quad (11b)$$

$\rho_2$  is plotted in figure 2 for selected examples, where  $\vec{r}_2$  is fixed at the  $X$ -axis (marked by a black spot) and  $\rho_2$  is considered as a function of  $\vec{r}_1$  moving in the  $X$ - $Y$  plane (only given in the upper plane due to the up-down symmetry). This figure confirms the ring structure suggested previously [7–10]. In particular,  $N_{out}$ , the number of electrons in the ring, is found to be strongly type dependent as we shall see. Nonetheless, for higher states, the structures are less constrained by symmetry because there are many choices for them (many possible ways of producing excitation). Therefore their structures are difficult to foresee; we will not discuss them in general. Incidentally, in contrast to the atoms, the dots do not contain a few valence electrons outside but a ring with many electrons. This is an interesting point.

Let us first inspect the  $L = 60$  states belonging to the type 1B. Both the 8- and 6-ring structures are favourable choices for this type; they are found to appear in  $(60)_1$  and  $(60)_2$  states, respectively, as shown in figures 2(a) and (b). These two states are crystal-like. The 7-ring structure is not expected because it is not a favourable choice.



**Figure 2.** Contour plots of  $\rho_2(\vec{r}_1, \vec{r}_{20})$  for a nine-electron dot as a function of  $\vec{r}_1$ .  $\hbar\omega_0 = 3$  meV is assumed. The fixed  $\vec{r}_{20}$  is marked by a black spot and the distance  $\vec{r}_{20}$  is given in each panel in units of  $\sqrt{\hbar/(m^*\omega_0)} = 194.7$  Å. The innermost contour (associated with the highest peak) is marked by a double line.



**Figure 3.** Contour plots of  $\rho_3$  of 9-electron ((a) and (b)) and 19-electron ((c) and (d)) dots. The two fixed electrons are marked by black spots.  $\hbar\omega_0 = 3$  meV is assumed. Different scales have been used for different states..

An example of the type 1A is the  $(36)_1$  state shown in figure 2(c), which is liquid-like. It is noted that a clear geometric structure arises from the coherent mixing of a number of basis functions. However, for the case of  $L = 36$ , there is only one basis function in the lowest Landau level. This basis function, the Laughlin wavefunction with  $j_o = 1$  (cf equation (7)), dominates the  $(36)_1$  state. Due to the features of this wavefunction and due to the lack of coherent mixing, the  $(36)_1$  state cannot possess a clear geometric feature. Nonetheless, the higher  $L = 36$  states do have clear ring structures (not shown here).

Figures 2(*d*) and (*e*) show examples of the type 3B; they have a clear 6-ring structure (which is the only favourable choice). To see more clearly the structure of the core,  $\rho_3$  for the  $(81)_1$  state is plotted in figure 3(*a*), where the core has a clear regular triangle shape.

Figure 2(*f*) shows an example of the type 3A. Both the 7- and 6-ring structures are favourable choices for this type; however, the 7-ring is lower in potential energy. Therefore, the first-state  $(99)_1$  chooses the 7-ring, as shown in the figure.

Figures 2(*g*)–(*j*) show examples of the type 4A; only the 7-ring structure is a favourable choice for this type. The  $(38)_3$ ,  $(50)_1$  states of this type are found to be 7-rings. It is noted that the  $L = 38$  states have only two basis functions in the lowest Landau level. Thus the  $(38)_1$  and  $(38)_2$  states are liquid-like (figures 2(*g*) and (*h*)) due to the lack of coherent mixing, whereas the  $(38)_3$  state is dominated by a number of basis functions in the second-lowest level; therefore a clear geometric structure emerges.

Figures 2(*k*)–(*n*) also show examples of the type 3B. The first-states plotted in figures 2(*k*) and (*l*) each have a clear 6-ring as expected. However, the  $L = 48$  and 96 states are not only 3(3) accessible but also 4(2) accessible, because they have  $n_2 \geq 2$  (cf rule 1). Owing to the 4(2) accessibility, a double-square structure was found in the  $(48)_2$  state as shown in figure 2(*m*), where each vertex of the smaller square is located at the mid-point of a side of the larger square. To show more clearly the double square,  $\rho_3$  of the  $(48)_2$  state is plotted in figure 3(*b*). On the other hand, although the 8(1) configuration is inaccessible to the type 3B, the  $(96)_2$  state was found to have an 8-ring structure (figure 2(*n*)). This implies that a higher 8(1)-inaccessible state might have an 8-ring structure. In this case a node will appear in the wavefunction in the centred-octagon configuration, and thereby a stronger internal motion (e.g., the relative oscillation of the electrons in the ring) is involved. In other words, 8(1)-inaccessible states can only possess excited 8-ring structure.

Figure 2(*o*) shows an example of the type 2B, where the  $(44)_1$  state is shown to have an 8-ring structure as expected.

Figure 2(*p*) shows an example of the type 2A. Both the 7-ring and 8-ring are favourable choices for this type; there is competition. Since the optimized potential energies of these two ring structures are very close to each other, which one would be the ‘winner’ is difficult to foresee. It turns out that the  $(52)_1$  state has an 8-ring structure (figure 2(*p*)). However, the first-states of this type are found to have a 7-ring structure if  $L \geq 148$ , because the 7-ring has a larger size and thereby a larger moment of inertia. For a state with a large  $L$ , the collective rotation energy is very large; therefore the structure with a larger moment of inertia would be chosen to reduce the rotation energy. This is a point requiring further clarification.

In addition to the above examples,  $N_{out}$ -values for the first-states are listed in table 3 (for  $44 \leq L \leq 73$ , the same range as in figure 1).

It is clear from the table that, for the first-states, the types determine to a great extent the numbers of electrons contained in the ring; thus the classification makes sense. There are three types, namely the 4A, 2B, and 3B, having only one favourable choice. For each first-state for these types the  $n$ -ring structure is just the favourable choice determined by the type. As an example, all the first-states of the type 4A under consideration have the 7-ring structure (the only favourable choice). However, there is an exception for the type 3B: instead of a 6-ring, the  $(72)_1$  state is a c.m.-excited version of the  $(71)_1$  state; this is a point requiring clarification.

There are four types each having more than one favourable choice, namely the types 1A, 2A, 3A, and 1B. In these cases a competition among the  $n$ -ring structures arises. It has been stated that the (optimized) potential energies of the 8- and 7-ring structures are the lowest, the 6-ring is the second lowest, while the 5- and 9-rings are remarkably higher. In the type 2A the 8-ring wins in the competition with the 7-ring when  $L$  is not so large; however the 7-ring wins when  $L \geq 148$ . In the type 3A the 7-ring wins in the competition with the 6-ring when  $L$  is

**Table 3.** Ring structures of the first-states with ( $44 \leq L \leq 73$ ). ( $n$ ) denotes an  $n$ -ring structure, an excited  $n$ -ring structure (with excited internal oscillation) is denoted by ( $\mathbf{n}$ ), ( $\bar{\phantom{n}}$ ) denotes an ambiguous structure, and (c.m.) denotes that the c.m. motion has been excited. The states associated with the plateaus in figure 1 are marked by asterisks.

Type	$L$
1A	
2A	52*(8)
3A	45*(9), 57*(7), 66*(7)
4A	50(7), 59(7), 64*(7), 71*(7), 73(7)
1B	60*(8)
2B	44(8), 68(8)
3B	48(6), 51(6), 54(6), 63(6), 69(6), 72(c.m.)
4B	46(c.m.), 47( $\bar{\phantom{n}}$ ), 49( $\bar{\phantom{n}}$ ), 53(c.m.), 55(5), 56( $\bar{\phantom{n}}$ ) 58(c.m.), 61(c.m.), 62(5), 65(c.m.), 67(7), 70(6)

not so large; however, the 6-ring wins when  $L \geq 141$ . In the type 1B the 8-ring wins in the competition with the 6-ring when  $L$  is not so large; however, the 6-ring might win when  $L$  is large enough.

When  $L$  gets larger and larger, collective rotation energy becomes more and more important, and thus a structure with a large moment of inertia becomes very attractive. It is not yet clear whether, when  $L$  is very large, other structures with moment of inertia larger than the ring structure (say, an ellipse) should be pursued. This is an interesting point.

It is noted that the angular momenta  $l_i$  of the electrons in the ring are usually much larger than those in the core. However, for the single-vortex state  $(45)_1$ , all the  $l_i$  are closely packed, ranging from  $l_i = 1$  to 9 (two  $l_i$  cannot be equal unless one electron jumps to a higher Landau level). Consequently, the ring and the core of the  $(45)_1$  state are very close to each other, so they in fact form a broad ring with the centre empty. Among the many first-states of different types that we have calculated, only the  $(45)_1$  state has such a special structure.

All the 7-, 8-, and 6-ring structures are not favourable choices for the type 4B. It is shown in the table that they either have an ambiguous structure, or a 5-ring structure (higher in potential energy), or have the c.m. excited. However, the members with larger  $L$  may have an  $n$ -ring structure ( $6 \leq n \leq 8$ ). It was found that meanwhile the wavefunction is not smoothly distributed in the ring but contains a node (e.g., if one electron locates at the centre and seven electrons locate at the vertices of an octagon, then the empty vertex is a node of the wavefunction). This implies that only the excited  $n$ -ring structures can emerge in this type.

The above investigation confirms the great effect of symmetry, and the classification scheme works almost perfectly.

## 5. A general dot and a summary

For a general  $N$ -electron dot, as a standard procedure, first we have to work out which configurations are important (lower in potential energy); then we have to study the accessibility of these configurations, i.e., to which  $L$ -series they are accessible. For an  $m(k)$  configuration the accessibility can be clarified by making use of rules 1–3. For other cases (where  $N_{out}$  and  $N_{in}$  do not have a common factor), we have to study how the subsystems are constrained by symmetry. According to the accessibility, the states can be classified, and we can understand which structures are favourable choices for a specific type of state.

When  $N = 6$ , the 5(1), 3(2), 2(3) configurations should be considered (In this way, the 6(1) configuration has been automatically taken into account, because once both the 3(2) and

2(3) configurations are accessible to an  $L$ -series, the 6(1) configuration is also accessible due to the rule 3). The type able to get access to all of these  $m(k)$  has  $n_2 = 0, n_3 \geq 1, n_5 \geq 1, n_i \geq 0$ , which is just the intersection of  $\{L \equiv 0 \pmod{5}\}$  and  $\{L \equiv 3 \pmod{6}\}$ . It is well known that the states with these  $L$  are candidates for being ground states [4].

When  $N = 19$ , the 12-ring structure is important. On the other hand, there is a highly symmetric 6(3) configuration composed of 24 regular triangles (i.e., each inner particle keeps equidistant from all its neighbours). This configuration is a minimum of the potential energy; the radii of the three homocentric circles are equal to 1,  $\sqrt{3}$ , and 2. Since the radii of the two outer circles are close to each other, the wavefunction of a 12-ring structure [6] will be mainly distributed in a domain containing the 6(3) configuration. Therefore, the 6(3) accessibility is crucial to whether the 12-ring is a favourable choice. From rule 2 we know that a 6(3)-accessible state has  $n_2 = 0$  and  $n_3 \geq 1$ . Examples of  $\rho_3$  belonging to and not belong to this type, respectively, are shown in figures 3(c) and (d). A very clear 12-ring structure is shown in figure 3(c). A more detailed discussion on  $N > 9$  dots will be given elsewhere.

In summary, a classification scheme has been proposed in this paper: the states are in essence classified according to their ability to get access to symmetric configurations. In this scheme, each type has its own preference for structures. As an example, the states of a nine-electron dot have been classified in detail. Furthermore, an exact diagonalization of the Hamiltonian has been performed. Since higher Landau levels have been included, accurate (in the qualitative sense) eigenstates have been obtained. In addition to  $\rho_2$ -values, which have been calculated quite often in the literature,  $\rho_3$ -values (which have hardly ever been calculated for dots before) have also been calculated to help in the analysis. The great effect of symmetry and the plausibility of the classification have been confirmed by the numerical results. Thus, the electronic structures can be more or less foreseen. In particular, the ring structures of the first-states can be predicted accurately to a great extent. There is a similarity existing among the  $L$ -series of the same type. The symmetry background underlying the stability of the Laughlin wavefunction and the single-vortex states has also been discussed.

A radically different approach to the classification of states is based on the composite fermion model [22]. This model works very well when the number of electrons  $N$  is small. When  $N$  is larger, some discrepancies have been reported (e.g., this model does not predict the magic angular momenta  $L = 40$  and  $50$  of the six-electron system). The validity of this model is not clear when  $N$  is even larger. On the other hand, our approach is mainly based on symmetry considerations and therefore is basically model independent. Also, our approach is designed for dots with a larger  $N$ . Nonetheless, to confirm the validity of our approach for  $N > 9$  systems, a detailed investigation for these systems is necessary. This is under consideration.

Although only the case of  $S = N/2$  is considered in this paper, the idea and the formalism (cf equations (2) and (3)) are general; thus the consideration can be directly generalized to the case of  $S \neq N/2$ . In fact, the symmetry constraint is a universal concept, which is embodied by the  $m(k)$  accessibility for all kinds of two-dimensional systems and subsystems with a centre. The introduction of this concept will lead to a better understanding of these systems.

Specifically, the above discussion can be generalized to cover bosonic systems, to deal with the Bose–Einstein condensates. For bosonic systems, equation (6) is replaced by

$$\exp[i\pi(2^{n_2+1}-m_2 3^{n_3-m_3} 5^{n_5-m_5} \dots)] = 1. \quad (12)$$

Therefore the states able to get access to an  $m(k)$  configuration should have  $n_i \geq m_i$ , or should have  $L = jm$ . With this in mind, the magic angular momenta found in a two-dimensional rotating bosonic system [23] can be explained. For example, the magic number  $L = 42$  of an  $N = 7$  system arises from 7(1) and 6(1) accessibility, the magic number  $L = 56$  of an  $N = 8$

system arises from 8(1) and 7(1) accessibility, the magic number  $L = 42$  of an  $N = 9$  system arises from 8(1) and 3(3) accessibility, the magic number  $L = 90$  of an  $N = 10$  system arises from 9(1), 5(2), and 3(3) accessibility, and so on.

Number theory and quantum mechanics are two previously unrelated areas of science. Here we show a relation between them—namely, that of the primary classification of electronic structures and the ultimate resolution of  $L$  as an integer.

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